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We derive two coupled non-linear evolution equations corresponding to the truncation of the Balitsky infinite hierarchy of saturation equations after inclusion of dipole-dipole correlations, i.e. one step beyond the Balitsky-Kovchegov (BK) equation. We exhibit an exact solution for maximal correlation which still satisfies the same asymptotic geometric scaling as BK but with the S-matrix going to 1/2 (instead of 0) in the full saturation region.

1. In perturbative QCD, parton saturation, i.e. the modification of the distribution of quarks and gluon distributions in a target, is known to lead to an infinite set of coupled evolution equations in energy for the correlation functions of multiple Wilson lines [1]. This set of equations is expected to be equivalent to the Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) functional equation [2]. In the approximation where correlation functions for more than two Wilson lines factorize, i.e. $\langle \text{tr } U^2 \text{tr } U^2 \rangle \sim \langle \text{tr } U^2 \rangle \langle \text{tr } U^2 \rangle$, the problem reduces to a unique non-linear Balitsky-Kovchegov (BK) equation [1, 3] for the dipole density. This amounts to considering a large- N_c limit of independent dipole-target collisions, e.g. on a large nucleus. Recently it was shown that the translation-invariant (no impact parameter b -dependence) non-linear BK equation lies in the universality class of the Fisher and Kolmogorov-Petrovsky-Piscounov (F-KPP) equation [4], leading to asymptotic traveling wave solutions in the transition region to saturation. This provides a mathematical realization of the phenomenologically motivated [5] geometrical scaling. In addition, the BK equation leads to an S-matrix element which goes to 0 in the full saturation region. Note that, if one assumes that translation invariance is a good approximation for scattering near $b = 0$, this amounts to the black disk limit $\sigma_{el}/\sigma_{tot}(b \sim 0) \rightarrow 1/2$.

The problem we want to address is the study of QCD saturation including the effects of nontrivial two-dipole correlations. To this end we have to proceed one step further in the hierarchy of Balitsky equations [1]. Our main observation is that one can perform a truncation of this set at the level of two-dipole correlations, as follows:

- We keep only dipole-like terms (i.e. we neglect higher multipoint traces like $\langle \text{tr } U^4 \rangle$ and $\langle \text{tr } U^6 \rangle$).
- We keep full two-dipole correlations and only neglect independent three- and higher dipole correlations.

In this paper we derive the general closed set of two non-linear equations for the dipole density and the dipole-dipole correlation function, see equations (9,10) below. Moreover, using the translational invariance hypothesis (no b -dependence), we find a particularly convenient form of these equations in a Fourier transform representation see equations (13,14). We then exhibit a solution in terms of a modified BK equation when assuming *maximal* correlation, i.e. when the correlation stays independent of the separation distance between the two dipoles see (21,22). We analyze the solutions, examine the modifications with respect to BK ones and discuss the prospects of our study. As an aside we note that some hierarchies of evolution equations have been studied in a statistical physics context in [6].

2. Let us briefly recall the standard derivation of the BK equation in the Balitsky's framework [1]. One defines the dipole operator

$$\mathcal{D}_{ij} = \text{tr } U_{x_i} U_{x_j}^\dagger, \quad (1)$$

where $x_{i,j}$ are the transverse coordinates of the end point quark and antiquark of a QCD dipole. One calculates its evolution with rapidity $\partial/\partial Y$ using contractions between pairs of U 's and virtual corrections for each U (see e.g. eqs. (119,120) in [1].)

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Within the infinite set of saturation equations, the one for the evolution of the dipole operator [7] expectation value with rapidity reads:

$$\frac{\partial}{\partial Y} \langle \mathcal{D}_{01} \rangle = \frac{g^2}{8\pi^3} \int d^2 x_2 [\langle \mathcal{D}_{02} \mathcal{D}_{21} \rangle - N_c \langle \mathcal{D}_{01} \rangle] K_{021} \quad (2)$$

where the kernel K_{021} is

$$K_{021} = \frac{x_{01}^2}{x_{02}^2 x_{21}^2}, \quad (3)$$

i.e. the real dipole splitting part in the BFKL kernel [8]. Let us now decompose the expectation values into 1-point functions and connected correlation functions through

$$\langle \mathcal{D}_{02} \rangle = d_{02} \quad (4)$$

$$\langle \mathcal{D}_{02} \mathcal{D}_{21} \rangle - \langle \mathcal{D}_{02} \rangle \langle \mathcal{D}_{21} \rangle = d_{0221} \quad (5)$$

The BK equation is obtained by *neglecting* d_{0221} , i.e. the connected part of the two-dipole correlation function, with two coincident points at x_2 .

Our first goal is to keep now this term, and derive an equation for this quantity. In order to obtain a well-defined *closed* set of equations, it turns out to be necessary to write an equation for the correlation of arbitrary two dipoles, not constrained to have one point in common. We thus consider more generally $\partial \langle \mathcal{D}_{02} \mathcal{D}_{2'1} \rangle / \partial Y$ and perform all contractions between the unitary matrices.

For the truncation, we proceed as follows. All single (1-point) and double contractions within the same dipole give essentially two copies of (2) with the other dipole included as a spectator. To this we have to add contractions between U 's belonging to different dipoles. These generate only non-dipole terms of the form $\langle \text{tr } U^6 \rangle$ and $\langle \text{tr } U^4 \rangle$ which we neglect by assumption. The resulting equation reads

$$\frac{\partial}{\partial Y} \langle \mathcal{D}_{02} \mathcal{D}_{2'1} \rangle = \frac{g^2}{8\pi^3} \int d^2 x_3 [\langle \mathcal{D}_{03} \mathcal{D}_{32} \mathcal{D}_{2'1} \rangle - N_c \langle \mathcal{D}_{02} \mathcal{D}_{2'1} \rangle] K_{032} + [\langle \mathcal{D}_{2'3} \mathcal{D}_{31} \mathcal{D}_{02} \rangle - N_c \langle \mathcal{D}_{02} \mathcal{D}_{2'1} \rangle] K_{132'} . \quad (6)$$

When $2 = 2'$ this reduces to the expression in [1] up to the neglected $\langle \text{tr } U^6 \rangle$ and $\langle \text{tr } U^4 \rangle$ terms. Keeping track of all two-dipole correlations, the expectation values are evaluated giving

$$\langle \mathcal{D}_{03} \mathcal{D}_{32} \mathcal{D}_{2'1} \rangle = d_{03} d_{32} d_{2'1} + d_{0332} d_{2'1} + d_{322'1} d_{03} + d_{032'1} d_{32} . \quad (7)$$

It is important to notice the appearance here of two-dipole correlation functions with four non coinciding points even if we would start with $2 = 2'$, due to the contractions between the first and third \mathcal{D} operators. This justifies the need to get the corresponding 4-point evolution equations. Indeed, neglecting this term would lead to spurious divergences in the equation. In this way we get:

$$\frac{\partial}{\partial Y} d_{022'1} = \frac{g^2}{8\pi^3} \int d^2 x_3 \{ d_{03} d_{322'1} + d_{32} d_{032'1} - N_c d_{022'1} \} K_{032} + \{ (0, 1) \iff (2', 2) \} \quad (8)$$

Going from the Wilson line correlation traces d to dipole densities ($d_{01} = N_c(1 - N_{01})$ and $d_{022'1} = N_c^2 N_{022'1}$) we get the final closed set of equations:

$$\frac{\partial}{\partial Y} N_{022'1} = \frac{g^2 N_c}{8\pi^3} \int d^2 x_3 \{ [N_{322'1} + N_{032'1} - N_{022'1}] K_{032} - [N_{03} N_{322'1} + N_{32} N_{032'1}] K_{032} \} + \{ (0, 1) \iff (2', 2) \} \quad (9)$$

$$\frac{\partial}{\partial Y} N_{01} = \frac{g^2 N_c}{8\pi^3} \int d^2 x_2 [N_{02} + N_{21} - N_{01} - N_{02} N_{21} - N_{0221}] K_{021} . \quad (10)$$

3. It is well known [3] that after imposing translational invariance (i.e. restricting to impact-parameter independent regime) and going over to momentum space the BK equation greatly simplifies. We will perform an analogous procedure for our set of equations (9)-(10). Let us introduce the following 2-dimensional Fourier transforms

$$N_{01} = x_{01}^2 \int \frac{d^2 k}{2\pi} e^{ik \cdot x_{01}} N_k \quad (11)$$

$$N_{022'1} = x_{02}^2 x_{2'1}^2 \int \frac{d^2 q}{2\pi} \frac{d^2 q'}{2\pi} d^2 Q e^{iq \cdot x_{02}} e^{iq' \cdot x_{2'1}} e^{iQ \cdot (x_{02'} + x_{21})} N_{qq', Q} \quad (12)$$

Here q and q' are 2-vectors conjugate to the dipole vectors, while Q is the variable dual to the 2-vector drawn between the center of masses of the two dipoles. The nonlinear terms now become local in momentum space, while the linear terms reduce essentially to the BFKL kernel [9]. We get finally:

$$\frac{\partial}{\partial Y} N_q = \frac{g^2 N_c}{4\pi^2} \{2\chi(-\partial_L) N_q - N_q^2 - \int d^2 Q N_{(q-Q)(q-Q),Q}\} \quad (13)$$

$$\frac{\partial}{\partial Y} N_{qq',Q} = \frac{g^2 N_c}{4\pi^2} \{2\chi(-\partial_L) N_{qq',Q} + 2\chi(-\partial_{L'}) N_{qq',Q} - (N_{q+Q} + N_{q-Q} + N_{q'+Q} + N_{q'-Q}) N_{qq',Q}\} , \quad (14)$$

where $L \equiv \log(q)$,

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \quad (15)$$

and $\chi(-\partial_L)$ is an integro-differential operator which may be defined [4] with the help of a formal series expansion around some given γ_0 between 0 and 1, *i.e.* for the principal branch of the function χ . Note that the integral term in the first equation (13) corresponds to the integration of a dipole-dipole correlator (5) with the kernel K_{021} over the separation distance of the two dipoles with one coinciding endpoint.

The initial conditions for correlations can be formulated in the variable Q by assuming a gaussian form $e^{-l_{corr}^2 Q^2}$ corresponding to a transverse correlation length l_{corr} . In order to illustrate the impact of these correlations on the problem, let us consider the limiting case of *maximal correlations* *i.e.* when the dipole-dipole correlations are local in Q ($l_{corr} \rightarrow \infty$):

$$N_{qq',Q} = N_{qq'} \delta^2(Q) . \quad (16)$$

Then the equations simplify and we get

$$\frac{\partial}{\partial Y} N_q = \frac{g^2 N_c}{4\pi^2} \{2\chi(-\partial_L) N_q - N_q^2 - N_{qq}\} \quad (17)$$

$$\frac{\partial}{\partial Y} N_{qq'} = \frac{g^2 N_c}{4\pi^2} \{2\chi(-\partial_L) N_{qq'} + 2\chi(-\partial_{L'}) N_{qq'} - 2(N_q + N_{q'}) N_{qq'}\} . \quad (18)$$

Remarkably enough this set of equations admits an exact solution (apart from the trivial solution $N_{qq'} = 0$) using the ansatz $N_{qq'} = \lambda N_q N_{q'}$. Note that this form may be quite plausible physically as typically these dipoles are separated by very large distances, therefore we do not expect strong dependence of the correlations on the *relative* sizes. Plugging the ansatz into equations (17)-(18) we obtain $\lambda = 1$ and find the following system

$$N_{qq'} = N_q N_{q'} \quad (19)$$

$$\frac{\partial}{\partial Y} N_q = \frac{g^2 N_c}{4\pi^2} \{2\chi(-\partial_L) N_q - \mathbf{2} N_q^2\} , \quad (20)$$

or equivalently in position space:

$$N_{022'1} = N_{02} N_{2'1} \quad (21)$$

$$\frac{\partial}{\partial Y} N_{01} = \frac{g^2 N_c}{8\pi^3} \int d^2 x_2 [N_{02} + N_{21} - N_{01} - \mathbf{2} N_{02} N_{21}] K_{021} . \quad (22)$$

In fact this solution can be extended to the full b -dependent set of equations (9,10).

It is important to note the factor $\mathbf{2}$ in front of the non-linear term which is the consequence of the correlations. Note that a modification of the BK equation of the type (22) but with $\mathbf{2}$ replaced by some constant has been proposed in [12] on phenomenological grounds to account for the effect of correlations in nuclei. Our approach gives a derivation of such a modification from the JIMWLK-Balitsky framework in the special limit of infinite range correlations and opens up a way to study corrections due to finite correlation length.

The solution of the equation (22) can be obtained from the solution of the *uncorrelated* BK equation through the relation

$$N_{01}(l_{corr} \rightarrow \infty) \equiv \frac{1}{2} N_{01}^{BK} . \quad (23)$$

The S matrix is consequently $S = 1 - N = 1 - \frac{1}{2} N^{BK}$ and satisfies the equation:

$$\frac{\partial}{\partial Y} S_{01} = \frac{g^2 N_c}{8\pi^3} \int d^2 x_2 [2(S_{02} - \frac{1}{2}) \times (S_{21} - \frac{1}{2}) - (S_{01} - \frac{1}{2})] K_{021} . \quad (24)$$

A consequence of the above equations is that the solution for the case with maximal correlations leads to a saturation regime where the S-matrix goes to $\frac{1}{2}$ instead of 0. Physically, it means that the ratio of the elastic over total cross-section (at small impact parameter at least) $\sigma_{el}/\sigma_{tot}(b \sim 0) \rightarrow \frac{1}{4}$ instead of $\frac{1}{2}$ for the BK equation. It is remarkable that the fixed point solution $S \rightarrow \frac{1}{2}$ saturates the so-called Pomplun bound [11], which states that $\sigma_{el} + \sigma_{dd} \leq \frac{1}{2}\sigma_{tot}$, where σ_{dd} is the contribution of inelastic diffractive channels to the total cross-section. Indeed, a simple calculation [11] gives $\sigma_{el} = \sigma_{dd} = \frac{1}{4}\sigma_{tot} = \frac{1}{4}$. It is interesting to notice that the phenomenological extraction of dipole-proton S-matrix values [13] seems to be consistent with $S \geq \frac{1}{2}$.

Another interesting aspect is that geometrical scaling and more generally, the transition to saturation is *not* expected to be modified, since it follows directly from the exact relation (23). It is yet another consequence of the universality properties of the traveling wave solutions [4]. The expectations from the mathematical properties of non-linear equations discussed in [4] show that the behaviour of the solution in that region are universal, i.e. independent of the precise form of the non-linear damping. Indeed, since equations (13)-(14) have a similar linear term as in the BK equation and lead to geometrical scaling both in the maximal correlation regime and in the regime of no correlations at all, therefore it is tempting to conjecture that geometrical scaling holds even when the assumption of maximal correlations is relaxed. This point certainly deserves further study.

4. In conclusion, we have shown that a consistent truncation can be performed on the infinite hierarchy [1] of Balitsky equations for n -point correlators of Wilson lines by keeping track of nontrivial two-dipole correlations. It results in a system of two coupled non-linear equations. It is interesting to note that the assumption of maximal correlations does not modify the traveling wave picture of geometrical scaling. In contrast, the fully saturated region is strongly modified, since the S-matrix admits a different limit.

Recently, it has been argued that a certain treatment of fluctuations beyond the BK equation may lead to deep modifications including strong geometrical scaling violations [14]. We do not find this phenomenon when including dipole-dipole correlators with maximal (very long-range) correlations. It would be interesting to look what is the behaviour of the solution for finite range correlations. A numerical simulation of our system of equations (13)-(14) seems feasible and could be helpful.

An important point to study is the dependence of the solutions on the initial conditions, in particular the stability of the fixed point solution for maximal correlations when one starts from generic initial conditions (apart from the initial condition with no correlations which trivially reduces to the standard BK equation). Interestingly enough it seems not out of reach to generalize the maximal correlation ansatz (21) to multipoint correlations in the JIMWLK context.

It is interesting to ask the question whether the connected dipole correlation functions could be directly related to physical observables. For instance, Fourier transforming back the function $N_{qq',Q}$ (see (14)) over q, q' leads to a relation with the interaction amplitude of 2 dipoles with the target. By crossing symmetry [15], some information can be obtained on the semi-inclusive *dipole*(r) \rightarrow *dipole*(r') + X scattering with transverse momentum Q on the target. This problem deserves more study, since it could give an interesting link with inelastic diffractive processes.

Acknowledgements. We thank Edmond Iancu for informing us of ref. [12] and Stéphane Munier for ref. [6]. RJ would like to thank Service de Physique Théorique, Saclay for hospitality when this work was carried out. RJ was supported in part by the KBN grant 2P03B08225 (2003-2006) and by the ECONET programme No 08155PC of the French Ministry of Foreign Affairs.

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- [1] I. Balitsky, Nucl. Phys. **B463** (1996) 99.
 - [2] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, Nucl. Phys. **B504** (1997) 415; Phys. Rev. **D59** (1999) 014014. H. Weigert, Nucl. Phys. **A703** (2002) 823. E. Iancu, A. Leonidov and L. McLerran, Nucl. Phys. **A692** (2001) 583; Phys. Lett. **B510** (2001) 133.
 - [3] Y. V. Kovchegov, Phys. Rev. **D60**, 034008 (1999); Phys. Rev. **D61**, 074018 (2000).
 - [4] S. Munier and R. Peschanski, Phys. Rev. Lett. **91**, 232001 (2003); Phys. Rev. D **D69**, 034008 (2004); hep-ph/0401215.
 - [5] A. M. Staśto, K. Golec-Biernat, and J. Kwiecinski, Phys. Rev. Lett. **86**, 596 (2001), hep-ph/0007192.
 - [6] G. Parisi and Y.-C. Zhang, J. Stat. Phys. **41**, 1 (1985).
 - [7] Here we use the “S-matrix” version of the equation. We pass to the “T-matrix” form below.
 - [8] L. N. Lipatov, Sov. J. Nucl. Phys. **23**, 338 (1976); E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, Sov. Phys. JETP **45**, 199 (1977); I. I. Balitsky and L. N. Lipatov, Sov. J. Nucl. Phys. **28**, 822 (1978).
 - [9] In the case of correlations this can be checked by noting that a convenient basis is provided by bilinears of the BFKL $SL(2, \mathbb{C})$ eigenfunctions [10]: $N_{022'1} = \sum n^{hh'}(z_\alpha, z_\beta) E^h(z_{0\alpha}, z_{2\alpha}) E^{h'}(z_{2'\beta}, z_{1\beta})$, acting with the linear kernel and only then going over to momentum representation.

- [10] L. N. Lipatov, Sov. Phys. JETP **63**, 904 (1986) [Zh. Eksp. Teor. Fiz. **90**, 1536 (1986)].
- [11] J. Pumplin, Phys. Rev. D **8**, 2899 (1973).
- [12] E. Levin and M. Lublinsky, Nucl. Phys. A **730** (2004) 191 [arXiv:hep-ph/0308279].
- [13] S. Munier, A. M. Stasto and A. H. Mueller, Nucl. Phys. B **603**, 427 (2001) [arXiv:hep-ph/0102291].
- [14] A. H. Mueller and A. I. Shoshi, arXiv:hep-ph/0402193.
- [15] A. H. Mueller, Phys. Rev. D **2**, 2963 (1970).